

The inversion problem for rational Bézier curves

Ana Marco, José-Javier Martínez*

*Departamento de Matemáticas, Universidad de Alcalá, Campus Universitario, 28871
Alcalá de Henares, Madrid, Spain*

Abstract

The inversion problem for rational Bézier curves is addressed by using resultant matrices for polynomials expressed in the Bernstein basis. The aim of the work is not to construct an inversion formula but finding the corresponding value of the parameter for each point on the curve. Since sometimes one has only an approximation of that point the use of the singular value decomposition, a key tool in numerical linear algebra, is shown to be adequate.

Keywords: Bernstein basis, resultant matrix, Bézier curve, inversion, singular value decomposition

1. Introduction

The inversion problem (given a point $P_0 = (x_0, y_0)$ on a curve C parametrized by $P(t) = (x(t), y(t))$), compute the value t_0 of the parameter t corresponding to this point P_0) is a classical problem in the area of Computer Aided Geometric Design (CAGD) which has applications, for instance, in computing the intersection points of two curves (Hoscheck and Lasser, 1993).

Two pioneering papers on this subject, which also study the problem of implicitization, are (Sederberg et al., 1984; Goldman et al., 1984), the use of resultant matrices being a fundamental tool in those papers. A nice recent survey of those and related problems can be found in (Sederberg and Zheng, 2002).

*Corresponding author

Email addresses: `ana.marco@uah.es` (Ana Marco), `jjavier.martinez@uah.es` (José-Javier Martínez)

Two recent alternative approaches to the inversion problem have been presented in (González-Vega and Rúa, 2009; Busé and D’Andrea, 2006).

Our aim in this paper is to solve numerically the inversion problem for the class of Bézier curves, a very popular type of curves in CAGD (Farin, 2002). Since for these curves the parametric equations are expressed in the Bernstein basis, it is convenient to construct the resultant matrices using the Bernstein basis directly, an idea already suggested in (Goldman et al., 1984).

A resultant matrix for polynomials expressed in the Bernstein basis is the so-called Bernstein-Bézout matrix, which has been used in (Bini and Gemignani, 2004) for designing a fast algorithm for computing the greatest common divisor of two polynomials expressed in the Bernstein basis, and then in (Marco and Martínez, 2007) for computing the implicit equation of a Bézier curve by means of interpolation. An algorithm for the fast computation of the Bézout resultant matrix for polynomials expressed in the monomial basis is introduced in (Chionh et al., 2002), and an algorithm for the fast computation of the Bernstein-Bézout resultant matrix (which we will use in our approach to the inversion problem) is presented in (Bini and Gemignani, 2004).

Another fundamental tool in our approach is the singular value decomposition (SVD). The SVD is one of the most valuable tools in numerical linear algebra and we will employ it in this work for computing an approximation of a basis of the nullspace of a Bernstein-Bézout matrix. The use of the SVD is specially appropriate when the matrix is not known exactly, what happens in our case when only an approximation of the point $P_0 = (x_0, y_0)$ is known. It must be observed that this is the usual situation, for example, when the inversion problem is utilized to solve problems of curve intersection (Marco and Martínez, 2004).

Related to this situation (the fact that the point is usually only close to, but not exactly on, the curve) in Section 10 of (Goldman et al., 1984) the problem of the geometric interpretation of *inversion formulae* was briefly addressed, and this problem was later deeply studied in (Wang and Joe, 1995).

It is important to point out that our approach does not carry out any transformation between Bernstein and power (monomial) basis, because this conversion could involve a greater loss of accuracy (Farouki, 1991). In this sense, in (Bini and Gemignani, 2004) is indicated that *for numerical computations involving polynomials in Bernstein form it is essential to consider algorithms which express all intermediate results using this form only*.

The rest of the paper is organized as follows. In Section 2 some basic

results on the Bernstein-Bézout matrix and the SVD are presented. In Section 3 our procedure for solving the inversion problem for a Bézier curve is introduced, leaving for Section 4 its analysis in the more general case of rational curves expressed in the Bernstein basis. Section 5 contains some numerical experiments that illustrate the performance of our approach, and finally Section 6 is devoted to conclusions.

2. Some basic results

In this section we recall some basic results on two fundamental tools that we will use in this work: the Bernstein-Bézout matrix and the singular value decomposition.

2.1. The Bernstein-Bézout matrix

The Bernstein-Bézout matrix is a resultant matrix for polynomials expressed in the Bernstein basis. It can be defined in an analogous way to the definition of the Bézout resultant matrix for polynomials expressed in the monomial basis (Sederberg and Zheng, 2002).

Let $p(t) = p_0\beta_0^{(n)}(t) + p_1\beta_1^{(n)}(t) + \cdots + p_n\beta_n^{(n)}(t)$ y $q(t) = q_0\beta_0^{(n)}(t) + q_1\beta_1^{(n)}(t) + \cdots + q_n\beta_n^{(n)}(t)$ be two polynomials expressed in the Bernstein basis

$$\mathcal{B}_n = \{\beta_i^{(n)}(t) = \binom{n}{i}(1-t)^{n-i}t^i, \quad i = 0, \dots, n\}.$$

The *Bernstein-Bézout matrix* $B = (b_{i,j}) \in R^{n \times n}$ of $p(t)$ and $q(t)$ is defined by

$$\frac{p(t)q(s) - p(s)q(t)}{t - s} = \sum_{i,j=1}^n b_{i,j} \beta_{i-1}^{(n-1)}(t) \beta_{j-1}^{(n-1)}(s),$$

which can be equivalently rewritten as

$$\frac{p(t)q(s) - p(s)q(t)}{t - s} = \begin{pmatrix} \beta_0^{(n-1)}(s) & \cdots & \beta_{n-1}^{(n-1)}(s) \end{pmatrix} B \begin{pmatrix} \beta_0^{(n-1)}(t) \\ \vdots \\ \beta_{n-1}^{(n-1)}(t) \end{pmatrix}.$$

The following result is a direct consequence of the definition of the Bernstein-Bézout matrix:

Theorem 1. *Let $p(t)$ and $q(t)$ be two polynomials expressed in the Bernstein basis \mathcal{B}_n , and B be the Bernstein-Bézout matrix of $p(t)$ and $q(t)$. If t_0 is a common root of $p(t)$ and $q(t)$, then $\det(B) = 0$.*

A fast algorithm for computing the entries of the Bernstein-Bézout matrix (an algorithm which we will use in this work) has been presented in (Bini and Gemignani, 2004). Since, for reasons explained in Section 5, we will use for the examples of that section the symbolic computation system *Maple* for the computation of the Bernstein-Bézout matrix, we include here the the Bini and Gemignani algorithm written in the *Maple* language:

```

for i from 1 to n do
  B[i,1] := (n/i)*(p[i]*q[0]-p[0]*q[i]);
od;

for j from 1 to n-1 do
  B[n,j+1] := (n/(n-j))*(p[n]*q[j]-p[j]*q[n])
od;

for j from 1 to n-1 do
  for i from 1 to n-1 do
    B[i,j+1] := (n^2/(i*(n-j)))*(p[i]*q[j]-p[j]*q[i])
      + ((j*(n-i))/(i*(n-j)))*B[i+1,j];
  od;
od;

```

2.2. The Singular Value Decomposition

Let us now recall the concept of singular value decomposition (SVD):

Given an $m \times n$ real matrix A , there exist orthogonal matrices U (of order m) and V (of order n), and a diagonal matrix Σ (of size $m \times n$) such that

$$A = U\Sigma V^T.$$

This factorization of A is called the *singular value decomposition (SVD)* of A .

The r (with $r \leq m, n$) nonzero diagonal entries of Σ are the *singular values* of A (i.e. the positive square roots of the eigenvalues of $A^T A$). If there are r (nonzero) singular values, then r is the *rank* of A .

But we also obtain a very important additional advantage from the computation of the SVD of a matrix A of size $m \times n$ (including V , not only Σ): *the last $n - r$ columns of V form a basis of the nullspace of A* (see (Strang, 1988; Demmel, 1997)).

The SVD provides the best way of estimating the rank of a matrix whose entries are floating point numbers, specially in the presence of round-off errors. This fact was clearly stated in a historical paper by Golub and Kahan (Golub and Kahan, 1965), where we can read the following sentences: “In the past the conventional way to determine the rank of A was to convert A to a row-echelon form... But in floating-point calculations it may not be so easy to decide whether some number is effectively zero or not... *In other words, without looking explicitly at the singular values there seems to be no satisfactory way to assign rank to A* ”.

Good general references in connection with the computation of the SVD of a matrix are (Golub and Van Loan, 1996) and (Demmel, 1997).

3. The inversion problem

Let $P(t) = (x(t), y(t))$ be a proper parametrization of a plane rational Bézier curve C given by:

$$x(t) = \frac{w_0 a_0 \beta_0^{(n)}(t) + w_1 a_1 \beta_1^{(n)}(t) + \cdots + w_n a_n \beta_n^{(n)}(t)}{w_0 \beta_0^{(n)}(t) + w_1 \beta_1^{(n)}(t) + \cdots + w_n \beta_n^{(n)}(t)},$$

$$y(t) = \frac{w_0 b_0 \beta_0^{(n)}(t) + w_1 b_1 \beta_1^{(n)}(t) + \cdots + w_n b_n \beta_n^{(n)}(t)}{w_0 \beta_0^{(n)}(t) + w_1 \beta_1^{(n)}(t) + \cdots + w_n \beta_n^{(n)}(t)},$$

$t \in [0, 1]$.

A parametrization $P(t) = (x(t), y(t))$ of a curve C is said to be *proper* if every point on C except a finite number of exceptional points is generated by exactly one value of the parameter t . Since every rational curve has a proper parametrization, we can assume that the considered parametrization is proper. Several results on the properness of curve parametrizations can be found in (Sendra and Winkler, 2001).

Our aim in this section is to solve the inversion problem for the case of a non exceptional given point $P_0 = (x_0, y_0) \in C$, where C is a Bézier curve properly parametrized by $P(t)$, that is to say, to compute the single value t_0 such that $P(t_0) = P_0$. The following result will be essential in our solution to this inversion problem.

Theorem 2. *Let $p(t)$ and $q(t)$ be two polynomials expressed in the Bernstein basis \mathcal{B}_n . If t_0 is a common root of $p(t)$ and $q(t)$, then the vector $(\beta_0^{(n-1)}(t_0), \beta_1^{(n-1)}(t_0), \dots, \beta_{n-1}^{(n-1)}(t_0))$ is in the nullspace of the Bernstein-Bézout matrix of $p(t)$ and $q(t)$.*

PROOF. Let B be the Bernstein-Bézout matrix of $p(t)$ and $q(t)$, \widehat{B} be the Bézout matrix of the polynomials $p(t)$ and $q(t)$ expressed in the monomial basis $\{1, t, t^2, \dots, t^n\}$, and N be the lower triangular matrix of change of basis from the Bernstein basis \mathcal{B}_{n-1} to the power basis $\{1, t, t^2, \dots, t^{n-1}\}$. The following relationship is satisfied (Bini and Gemignani, 2004):

$$\widehat{B} = NBN^T.$$

Let t_0 be a common root of $p(t)$ and $q(t)$, then

$$\widehat{B} \begin{pmatrix} 1 \\ t_0 \\ t_0^2 \\ \vdots \\ t_0^{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(see (Sederberg and Zheng, 2002)). In this way

$$NBN^T \begin{pmatrix} 1 \\ t_0 \\ t_0^2 \\ \vdots \\ t_0^{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and, as N is a regular matrix

$$BN^T \begin{pmatrix} 1 \\ t_0 \\ t_0^2 \\ \vdots \\ t_0^{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore the vector

$$N^T \begin{pmatrix} 1 \\ t_0 \\ t_0^2 \\ \vdots \\ t_0^{n-1} \end{pmatrix}$$

belongs to the nullspace of the Bernstein-Bézout matrix B . Taking into account that N is the matrix of change of basis from the Bernstein basis \mathcal{B}_{n-1} to the power basis $\{1, t, t^2, \dots, t^{n-1}\}$

$$N^T \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{n-1} \end{pmatrix} = \begin{pmatrix} \beta_0^{(n-1)}(t) \\ \beta_1^{(n-1)}(t) \\ \beta_2^{(n-1)}(t) \\ \vdots \\ \beta_{n-1}^{(n-1)}(t) \end{pmatrix}$$

and in consequence the vector $(\beta_0^{(n-1)}(t_0), \beta_1^{(n-1)}(t_0), \dots, \beta_{n-1}^{(n-1)}(t_0))$ is in the nullspace of B , the Bernstein-Bézout matrix of $p(t)$ and $q(t)$.

From now on, we will consider

$$\begin{aligned} p(t) &= w_0 a_0 \beta_0^{(n)}(t) + w_1 a_1 \beta_1^{(n)}(t) + \dots + w_n a_n \beta_n^{(n)}(t) \\ &\quad - x_0 (w_0 \beta_0^{(n)}(t) + w_1 \beta_1^{(n)}(t) + \dots + w_n \beta_n^{(n)}(t)), \end{aligned}$$

$$\begin{aligned} q(t) &= w_0 b_0 \beta_0^{(n)}(t) + w_1 b_1 \beta_1^{(n)}(t) + \dots + w_n b_n \beta_n^{(n)}(t) \\ &\quad - y_0 (w_0 \beta_0^{(n)}(t) + w_1 \beta_1^{(n)}(t) + \dots + w_n \beta_n^{(n)}(t)), \end{aligned}$$

and let B be the Bernstein-Bézout matrix of these two polynomials $p(t)$ and $q(t)$.

Taking into account Theorem 2, the vector $(\beta_0^{(n-1)}(t_0), \beta_1^{(n-1)}(t_0), \beta_2^{(n-1)}(t_0), \dots, \beta_{n-1}^{(n-1)}(t_0))$ is in the nullspace of B . Moreover, as $P_0 = (x_0, y_0)$ is a non exceptional point of C , which is properly parametrized by $P(t)$, the rank of B is $n-1$ and therefore the vector $(\beta_0^{(n-1)}(t_0), \beta_1^{(n-1)}(t_0), \dots, \beta_{n-1}^{(n-1)}(t_0))$ is a basis of the nullspace of B .

Let us observe that it is very usual not to know the point P_0 exactly. In this way, considering the results included in Section 2.2, it is convenient to

compute an approximation of a basis of the nullspace of B by using the SVD. Naturally, we do not have the exact matrix B but an approximation of B that we will denote by \tilde{B} . A theoretical study of the application of SVD to the computation of an approximation of the nullspace can be seen in Section 1 of Chapter 5 of Strang (1988).

As the matrix V of the SVD of \tilde{B} is orthogonal, the last column of V gives us an approximation of a multiple of the vector $(\beta_0^{(n-1)}(t_0), \beta_1^{(n-1)}(t_0), \dots, \beta_{n-1}^{(n-1)}(t_0))$ with Euclidean norm equal to 1:

$$(z_0, z_1, \dots, z_{n-1}) = \alpha(\beta_0^{(n-1)}(t_0), \beta_1^{(n-1)}(t_0), \dots, \beta_{n-1}^{(n-1)}(t_0)).$$

Since

$$\frac{z_i}{z_{i-1}} = \frac{\beta_i^{(n-1)}(t)}{\beta_{i-1}^{(n-1)}(t)} = \frac{\binom{n-1}{i}(1-t)^{n-1-i}t^i}{\binom{n-1}{i-1}(1-t)^{n-1-(i-1)}t^{i-1}} = \frac{(n-i)t}{i(1-t)}$$

we obtain the value of the parameter t we are interested in computing:

$$t_0 = \frac{iz_i}{iz_i + (n-i)z_{i-1}} \quad i = 1, 2, \dots, n-1.$$

4. A more general situation

In this section we show how our procedure can also be used for the case of a non Bézier rational curve C properly parametrized by $P(t) = (x(t), y(t))$, where $x(t) = \frac{u_1(t)}{u_2(t)}$, $y(t) = \frac{u_3(t)}{u_4(t)}$ and $u_i(t)$ ($i = 1, \dots, 4$) are polynomials expressed in the Bernstein basis \mathcal{B}_n . In this situation some care must be taken when $\deg(\bar{p}) \neq \deg(\bar{q})$, where $\bar{p}(t) = u_1(t) - xu_2(t)$ and $\bar{q}(t) = u_3(t) - yu_4(t)$.

Given a polynomial $p(t)$ expressed in the Bernstein basis \mathcal{B}_n , we denote by $\deg(p)$ the *degree of $p(t)$ expressed in the power basis*. It must be noticed that $n \geq \deg(p)$ (see Section 3 in (Busé and Goldman, 2008)).

The following example (Marco and Martínez, 2007) serves to show what happens in this special situation:

Example 1. Let \mathcal{B}_4 be the Bernstein basis of the space of polynomials of degree less than or equal to 4, and let us consider the curve given by the parametric equations

$$x(t) = \frac{4\beta_0^{(4)}(t) + 4\beta_1^{(4)}(t) + 3\beta_2^{(4)}(t) + 3\beta_3^{(4)}(t) + 7\beta_4^{(4)}(t)}{\beta_0^{(4)}(t) + \beta_1^{(4)}(t) + \beta_2^{(4)}(t) + \beta_3^{(4)}(t) + 3\beta_4^{(4)}(t)},$$

$$y(t) = 2\beta_0^{(4)}(t) + 3\beta_1^{(4)}(t) + 3\beta_2^{(4)}(t) + 3\beta_3^{(4)}(t) + 4\beta_4^{(4)}(t).$$

In this case

$$\bar{p}(t) = (4-x)\beta_0^{(4)}(t) + (4-x)\beta_1^{(4)}(t) + (3-x)\beta_2^{(4)}(t) + (3-x)\beta_3^{(4)}(t) + (7-3x)\beta_4^{(4)}(t)$$

and

$$\bar{q}(t) = (2-y)\beta_0^{(4)}(t) + (3-y)\beta_1^{(4)}(t) + (3-y)\beta_2^{(4)}(t) + (3-y)\beta_3^{(4)}(t) + (4-y)\beta_4^{(4)}(t).$$

However, if we write \bar{p} and \bar{q} in the power basis we have

$$\bar{p}(t) = 4 - x - 6t^2 + 8t^3 + (-2x + 1)t^4$$

(a polynomial of degree 4 in t), while

$$\bar{q}(t) = 2 - y + 4t - 6t^2 + 4t^3,$$

a polynomial of degree 3 in t . In this example, our approach works perfectly when computing the parameter t_0 corresponding to every point $P_0 = (x_0, y_0) \in C$ such that $x_0 \neq \frac{1}{2}$, i.e., it works for every point $P_0 \in C$ except for the point $P_0 = (1/2, -3.0395517)$. The reason why we have problems when $x_0 = \frac{1}{2}$ is that $x_0 = \frac{1}{2}$ makes 0 the leading coefficient of $\bar{p}(t)$, and the nullspace of the corresponding Bernstein-Bézout matrix has dimension greater than 1. When this happens, a possible solution is to consider the Sylvester matrix of the polynomials $\bar{p}(t)$ (expressed in the Bernstein basis \mathcal{B}_4) and $\bar{q}(t)$ (expressed in the Bernstein basis \mathcal{B}_3) (Winkler and Goldman, 2003) instead of the Bernstein-Bézout matrix of $\bar{p}(t)$ and $\bar{q}(t)$ expressed in the Bernstein basis \mathcal{B}_4 , and then proceed in the same way as we have described for the Bernstein-Bézout matrix.

5. Numerical examples

In this section we include several examples illustrating the performance of our procedure. A detailed description of the way in which we proceed at each step of our approach is included in the first one.

Example 2. Let us consider a rational Bézier curve C given by

$$P(t) = (x(t), y(t)) = \frac{\sum_{i=0}^{15} w_i(a_i, b_i) \binom{15}{i} t^i (1-t)^{n-i}}{\sum_{i=0}^{15} w_i \binom{15}{i} t^i (1-t)^{n-i}},$$

where the points of the control polygon are

$$\begin{aligned} (a_0, b_0) &= (14, 14), & (a_1, b_1) &= (11, 15), & (a_2, b_2) &= (9, 15), & (a_3, b_3) &= (7, 15), \\ (a_4, b_4) &= (4, 14), & (a_5, b_5) &= (3, 12), & (a_6, b_6) &= (3, 10), & (a_7, b_7) &= (7, 8), \\ (a_8, b_8) &= (4, 6), & (a_9, b_9) &= (14, 4), & (a_{10}, b_{10}) &= (12, 2), & (a_{11}, b_{11}) &= (8, 2), \\ (a_{12}, b_{12}) &= (6, 2), & (a_{13}, b_{13}) &= (4, 3), & (a_{14}, b_{14}) &= (3, 4), & (a_{15}, b_{15}) &= (2, 5), \end{aligned}$$

and the weights are

$$\begin{aligned} w_0 &= 2, & w_1 &= 2, & w_2 &= 2, & w_3 &= 1, & w_4 &= 2, & w_5 &= 5, & w_6 &= 5, & w_7 &= 1, \\ w_8 &= 3, & w_9 &= 3, & w_{10} &= 3, & w_{11} &= 3, & w_{12} &= 2, & w_{13} &= 1, & w_{14} &= 1, & w_{15} &= 1. \end{aligned}$$

Let us consider $P_0 = (8.50665, 14.3420)$, a point in C that we know approximately (it is an approximation of the point corresponding to the value of the parameter $t = \frac{1}{7}$). We take $P_0 = (x_0 = \frac{850665}{10^5}, y_0 = \frac{143420}{10^4})$ and we construct the Bernstein-Béout matrix of $p(t)$ and $q(t)$ by means of the *Maple* code introduced in Section 2.1 using exact arithmetic. The reason way we use exact arithmetic instead of floating point arithmetic is the following: although the algorithm presented in (Bini and Gemignani, 2004) for computing the entries of the Bernstein-Béout matrix is fast (it has a computational cost of $O(n^2)$ arithmetic operations) this algorithm does not have high relative accuracy, and therefore it is possible that important errors appear in the computation of some entries of the matrix.

Once we have in *Maple* the approximation \tilde{B} of the Bernstein-Béout matrix of $p(t)$ and $q(t)$, we put it in MATLAB where we compute its singular value decomposition. We include here the last column of the matrix V , which give us an approximation $\{(z_0, z_1, \dots, z_{n-1})\}$ of the basis of the nullspace of

B :

$$V(:, 15) = \begin{pmatrix} -2.473682899590338e - 001 \\ -5.771889609913881e - 001 \\ -6.253061379846553e - 001 \\ -4.169368183026464e - 001 \\ -1.910461999315792e - 001 \\ -6.368830764276734e - 002 \\ -1.592182997982856e - 002 \\ -3.033481337354997e - 003 \\ -4.422976092589377e - 004 \\ -4.912824640160427e - 005 \\ -4.044767281991494e - 006 \\ -2.444466877812876e - 007 \\ -1.184192736731686e - 008 \\ -5.593200217925554e - 011 \\ -4.503186536883329e - 011 \end{pmatrix}.$$

In the computation of t_0 we use two consecutive components of the vector above chosen in the following way: we select the vector component which has the greatest absolute value, and then we take among the component before and after this the one that have the greatest absolute value. We proceed in this way because the components with greatest absolute value are usually the ones which are less sensitive to perturbations (Demmel et al., 2008). In this case we obtain

$$t_0 = \frac{2z_2}{2z_2 + 13z_1} = 1.428606867264249e - 001.$$

When we compare the value of t_0 we have obtained with the exact value of the parameter $t = \frac{1}{7}$ of the exact point

$$P = \left(x\left(\frac{1}{7}\right), y\left(\frac{1}{7}\right)\right) = \left(\frac{78193109744768}{9191995131007}, \frac{131831466405881}{9191995131007}\right) \in C,$$

corresponding to the given point $P_0 = (8.50665, 14.3420)$, we get that the relative error we made when using our procedure is

$$\frac{|1/7 - t_0|}{1/7} = 2.48e - 005.$$

Taking into account that we have started from an approximation P_0 of the point $P \in C$ with only five exact digits, we can assert that the relative error that we have obtained by using our approach is small.

Now we present a detailed description of the computations for the non Bézier curve presented in Example 1.

Example 3. Let us consider the curve C introduced in Example 1 and a point $P_0 = (x_0, y_0) \in C$ such that $x_0 \neq \frac{1}{2}$, for instance $P_0 = (3.5542169, 2.8148148)$, which is an approximation of the point corresponding to the value of the parameter $t_0 = 1/3$. We show how our procedure works for these kind of points (all the points of C except one).

If we consider $P_0 = (\frac{35542169}{10^7}, \frac{28148148}{10^7})$, the singular values corresponding to the matrix \tilde{B} are:

$$\begin{aligned}\sigma_1 &= 4.212191730287018e + 000, \\ \sigma_2 &= 2.075444341475023e + 000, \\ \sigma_3 &= 5.981428444978487e - 001, \\ \sigma_4 &= 3.357757839963324e - 008.\end{aligned}$$

As only one singular value ($\sigma_4 = 3.357757839963324e - 008$) can be considered 0, the nullspace of \tilde{B} has dimension 1 and therefore the last column of the matrix V of the SVD of \tilde{B}

$$V(:, 4) = \begin{pmatrix} 5.111012703997072e - 001 \\ 7.666518828223443e - 001 \\ 3.833259078513024e - 001 \\ 6.388763832491218e - 002 \end{pmatrix}$$

gives us an approximation of the nullspace of the exact Bernstein-Béout matrix B . Proceeding in the same way as in the previous example we obtain that

$$t_0 = \frac{z_1}{z_1 + 3z_0} = 3.333333267311144e - 001,$$

a good approximation of $t = \frac{1}{3}$ if we take into account that P_0 is an approximation of the point corresponding to the value of the parameter $t_0 = 1/3$ with only 7 exact digits.

Now we show what happens when we consider $P_0 = (1/2, -3.0395517)$. In this case the singular values corresponding to the approximate Bernstein-Béout matrix \tilde{B} are

$$\begin{aligned}\sigma_1 &= 7.650996902942649e + 001, \\ \sigma_2 &= 4.412324130498519e + 001, \\ \sigma_3 &= 8.892019775556218e - 009, \\ \sigma_4 &= 1.013508461498068e - 015.\end{aligned}$$

Since the singular values σ_3 and σ_4 can be considered 0, the nullspace of \tilde{B} has dimension 2, and therefore it is not possible to proceed by using the approach we have described in Section 3. As we have pointed out in Section 4, a possible solution for this situation could be to consider the Sylvester matrix of the polynomials $\bar{p}(t)$ (expressed in the Bernstein basis \mathcal{B}_4) and $\bar{q}(t)$ (expressed in the Bernstein basis \mathcal{B}_2) (Winkler and Goldman, 2003) instead of the Bernstein-Bézout matrix of $\bar{p}(t)$ and $\bar{q}(t)$ expressed in the Bernstein basis \mathcal{B}_4 , and then proceed in the same way as we have described for the Bernstein-Bézout matrix.

Finally we present an example with a Bézier curve of small degree.

Example 4. Let us consider the rational Bézier curve C of degree 3 given by

$$P(t) = (x(t), y(t)) = \frac{\sum_{i=0}^3 w_i(a_i, b_i) \binom{3}{i} t^i (1-t)^{n-i}}{\sum_{i=0}^3 w_i \binom{3}{i} t^i (1-t)^{n-i}},$$

where the points of the control polygon are

$$(a_0, b_0) = (1, 9), \quad (a_1, b_1) = (2, 1), \quad (a_2, b_2) = (5, 1), \quad (a_3, b_3) = (4, 1),$$

and the weights are

$$w_0 = 1, \quad w_1 = 2, \quad w_2 = 2, \quad w_3 = 1.$$

In this case, as the considered curve has small degree, it is easy to obtain (by using the *Maple* command `ffgausselim`) the following explicit expression for an inversion formula:

$$f(x, y) = \frac{81408x + 13824yx - 7680x^2 - 276552 - 1008y + 1080y^2}{-216432 - 21024y + 67200x + 12672yx - 6144x^2 - 2160y^2}.$$

The exact point in C corresponding to the parameter $t = \frac{1}{3}$ is $P = (\frac{8}{3}, \frac{109}{45})$. When we consider its approximation $P_0 = (2.66667, 2.42222)$ the value of t_0 that we get by means of our approach is

$$3.333339104290224e - 001,$$

while the value of t_0 we obtain by using in *Maple* the above inversion formula $f(x, y)$ for P_0 is

$$3.333319852915495e - 001.$$

These results show that, even in the cases we have an explicit inversion formula, the computation of the parameter t_0 by evaluating this function does not necessarily give better results than using the SVD.

6. Conclusions

In this work we have shown how the combination of classical algebraic geometry tools (resultant matrices) and numerical linear algebra tools (the singular value decomposition) is adequate for addressing the inversion problem for a rational Bézier curve, although the more general situation of rational non Bézier curves defined by polynomials in the Bernstein basis is also considered. The resultant matrices being used avoid the conversion between the Bernstein basis and the monomial basis.

It must be observed that the aim of our work was not the computation of an inversion formula, which is not an easy task for curves of high degree and must be approached by using different techniques. Instead we have considered the numerical problem of finding the parameter value corresponding to a given point on the curve, a point that usually in practice is not known exactly.

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